

Synchronization on complex networks with different sorts of communities

Ming Zhao^{1,*}, Tao Zhou^{1,2,†} and Bing-Hong Wang^{1,3‡}

¹*Department of Modern Physics, University of Science and Technology of China, Hefei Anhui, 230026, PR China*

²*Department of Physics, University of Fribourg, Chemin du Muse 3, CH-1700 Fribourg, Switzerland*

³*Shanghai Academy of System Science, Shanghai 200093, PR China*

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In this paper, inspired by the idea that many real networks are composed by different sorts of communities, we investigate the synchronization property of oscillators on such networks. We identify the communities by the intrinsic frequencies probability density $g(\omega)$ of Kuramoto oscillators. That is to say, communities in different sorts are functional different. For a network containing two sorts of communities, when the community strength is strong, only the oscillators in the same community synchronize. With the weakening of the community strength, an interesting phenomenon, *Community Grouping*, appears: although the global synchronization is not achieved, oscillators in the same sort of communities will synchronize. Global synchronization will appear with the further reducing of the community strength, and the oscillators will rotate around the average frequency.

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I. INTRODUCTION

Many social, physical and biological systems are often described by networks. The collective dynamical behaviors of those networks are simultaneously determined by the individual systems, the interactions between individual systems, and the network structure. Very recently, it is found that the structure of many large-scale real networks are neither regular nor random, but of some common and important characters, such as short average distance, large clustering coefficient and the power-law degree distribution [1]. Networks of such structure are called complex networks. Besides the characters mentioned above, many real-world networks have the so-called community structure [2, 3]. A community in a network is a set of nodes, where the edges inside are much denser than those connecting the nodes belong to this set and the rest of the network. Examples of community networks are numerous in biological, technical and social systems. It is common in society that people are divided into different groups by their sex, age, interests and so forth, and each group can be considered as a community.

One of the most significant aims of the studies on complex networks is to understand the effects of the network structure on the dynamical processes, among which the synchronization of coupled oscillators is the simplest but one of the most important ones. Synchronization phenomenon has been observed for hundreds of years and in a variety of field, including natural, physical, chemical and biological systems [4]. Because of the limitation of knowledge on network structure, the studies of synchronization are restricted to either on the regular lattices or on the random networks for a long time period. Recently,

with the pioneer works on small-world [5] and scale-free network models [6], much attention has been paid to the studies of synchronization on those complex networks. Soon, scientist found delightedly that oscillators on complex networks are much easier to synchronize than on regular lattice with the same size and density of edges [7, 8, 9, 10], which further stimulates their enthusiasm.

Up to now, the relationship between network structure and synchronizability [11, 12, 13, 14, 15, 16, 17] is well understood and many ways to enhance the network synchronizability [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30] have been proposed (see also the review article [31] and the references therein). Very recently, the synchronization properties of networks with community structure are investigated, it is found that the connecting pattern between communities have great effects on the synchronization process [32, 33], and the community structure will hinder the global synchronization of oscillators [34, 35]: the stronger the community structure the worse the global synchronizability. All the works mentioned above concentrate on the networks with only one sort of communities, i.e., all the communities are similar in both function and topology. However, in real world, many networks have communities with different sorts, an example in point is a friendship network of children in a U.S. school [36]. As shown in Fig. 3.4 of Ref. [36], there are four communities: white middle school students, white high school students, black middle school students, and black high school students. Clearly, the four communities can be divided into two sorts by race: white and black, and there are two communities in each sort. Or, those communities can be divided by age, as middle school and high school.

In this paper, with the help of Kuramoto model [37, 38, 39, 40, 41], we investigate the synchronization properties of complex networks with different sorts of communities. In our study, oscillators on different sorts of communities are identified by the natural frequencies probability density $g(\omega)$, namely, the intrinsic frequen-

*Electronic address: zhaom17@mail.ustc.edu.cn

†Electronic address: zhutou@ustc.edu

‡Electronic address: bhwang@ustc.edu.cn

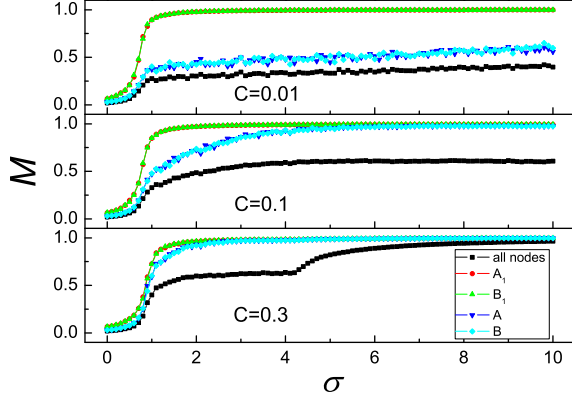


FIG. 1: (Color online) The relationship between order parameter M and the coupling strength σ for the network with two sorts of communities, and the number of communities in those two sorts are equal. Red circle, green triangle, blue up triangle and light blue diamond curves represent the order parameters of A_1 , B_1 , A and B , respectively. And the black square curve represents the order parameter of the whole network. Each point is obtained from 100 independent runs.

cies of nodes on different sorts of communities are taken differently.

This paper is organized as follows. In section II, the Kuramoto model and the order parameter are introduced. In section III, the network model we investigate will be described in detail. And also, we will give the simulation results of synchronization properties of Kuramoto oscillators on complex networks with different sorts of communities. The conclusion remarks are drawn in section IV.

II. KURAMOTO MODEL AND ORDER PARAMETER

In this paper, we use the coupled phase oscillators, Kuramoto model [37, 38, 39, 40, 41], to analyze the collective synchronization on complex networks. A modified Kuramoto model can be described by the coupled differential equation [35]:

$$\frac{d\phi_i}{dt} = \omega_i - \frac{\sigma}{k_i} \sum_{j \in \Lambda_i} \sin(\phi_i - \phi_j), \quad (1)$$

where ϕ_i , ω_i are the phase and the intrinsic frequency of node i , Λ_i is i 's neighbor set, and σ is the overall coupling strength. The intrinsic frequency ω_i is chosen from a probability density $g(\omega)$.

Usually, the intrinsic frequencies of different oscillators are assigned differently, if there are no couplings, all the oscillators will rotate independently, thus the phases of the oscillators are distributed almost uniformly in the

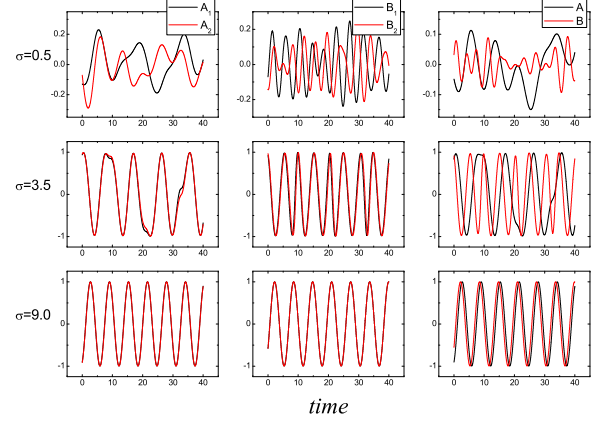


FIG. 2: (Color online) The averaged sine of phases of oscillators *vs.* time for community networks with $C = 0.3$ at different coupling strengths. There are two sorts with the same number of communities in the network.

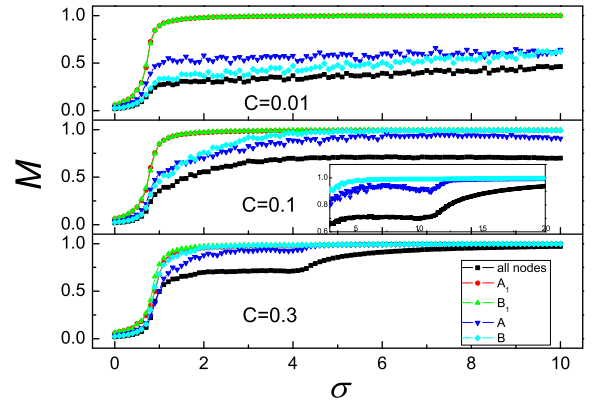


FIG. 3: (Color online) The relationship between order parameter M and the coupling strength σ for the network with two sorts of communities. Sort A and sort B contain 3 and 7 communities, respectively. The inset shows the change of order parameter in a larger scale.

interval $[0, 2\pi]$. With the increasing of the coupling strength, the oscillators will adhere to each other to some extent. At last, when the coupling strength reaches some critical point σ_c , collective synchronization of all oscillators emerges spontaneously, although the intrinsic frequencies of different oscillators are different.

To measure the synchronization phenomena, an order parameter M is introduced:

$$M = \left[\left\langle \left| \frac{1}{N'} \sum_{j=1}^{N'} e^{i\phi_j} \right| \right\rangle \right], \quad (2)$$

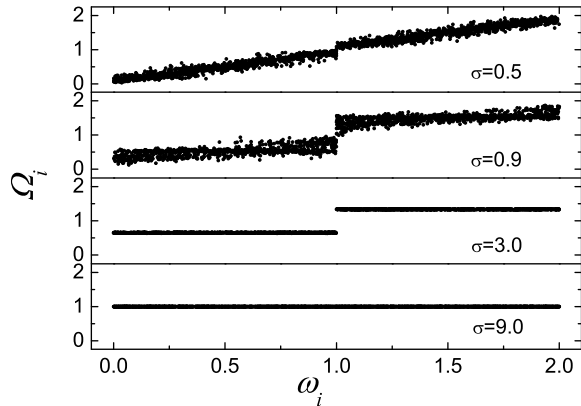


FIG. 4: The relationship between actual frequencies Ω_i and the intrinsic frequencies ω_i for the network, which is consisted of two sorts with equal number of communities.

where $\langle \dots \rangle$ and $[\dots]$ denote the average over time and over different configurations, respectively. N' is the number of nodes that are taken into account. Clearly, M is of order $1/\sqrt{N'}$ if the oscillators are completely uncoupled ($\sigma = 0$), and will approach 1 if they are all in the same phase. In our simulation, the dynamical equations integrate using the Runge-Kutta method with step size 0.01. The order parameters are averaged over 2000 time steps, excluding the former 2000 time steps to allow for the relaxation to a steady state. Note that, we consider not only the order parameter of all the oscillators in the network, but also investigate the synchronization of partial oscillators, such as the nodes belong to an individual community, or the nodes belong to a sort of communities. For different cases, the sum in Eq. (2) goes over the oscillators that are taken into account, and N' is taken accordingly.

III. NETWORK MODEL AND SIMULATION RESULTS

Our numerical simulations are based on the community network model in Refs. [35, 42]. The model starts from n community cores, each core contains m_0 fully connected nodes. Initially, there are no connections among different community cores. At each time step, there are n new nodes being added, each node will attach m edges to the existing nodes within the same community core, and simultaneously m' edges to the existing nodes outside this community core. The former are internal edges, and the latter are external edges. Similar to the evolutionary mechanism of Barabási-Albert networks [6], we assume the probability connecting to an existing node i is proportional to i 's degree k_i . Each community core will finally become a single community of size N_c , and

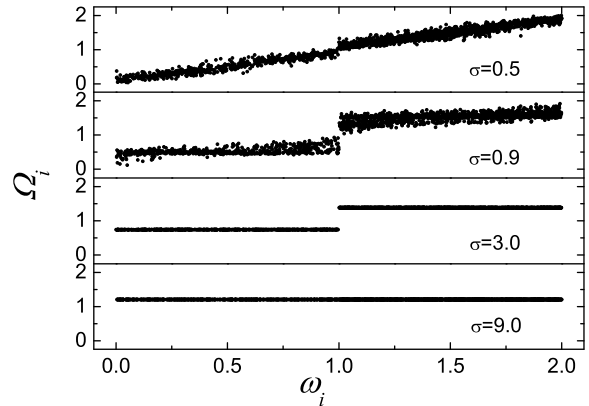


FIG. 5: The relationship between actual frequencies Ω_i and the intrinsic frequencies ω_i for the network, which is consisted of two sorts with different numbers of communities: sort A contains 3 communities, while sort B contains 7.

the network size is $N = nN_c$. By using the rate equation (similar to the analytical approaches used in Refs. [43, 44]), one can easily obtain the degree distribution of the whole network, $p(k) \propto k^{-3}$. We simply take the proportion of the external edges, $C = \frac{m'}{m+m'}$, to measure the community strength. Clearly, a smaller C corresponds to sparser external edges thus a stronger community structure.

In our simulation, a network of 10 communities is taken into account, and each community contains 200 nodes, thus there are totally 2000 nodes in the network. The average degree is set as $\bar{k} = 6$. The communities are divided into two sorts A and B, and each sort contains 5 communities, which are labeled as A_1, \dots, A_5 and B_1, \dots, B_5 , respectively. In fact, all the ten communities have similar structure, but the intrinsic frequencies of nodes in A are assigned randomly in $[0, 1]$, while those in B are assigned in $[1, 2]$.

To get the synchronization property of this kind of networks, we investigate how the order parameters in three scales, a community, a sort of communities and the whole network, change with the coupling strength. Figure 1 shows the simulation results. From top to bottom, C increases from 0.01, 0.1 to 0.3, corresponding to weaker and weaker community structure. In the case $C = 0.01$, with the increasing of coupling strength, the order parameter of individual community soon reaches 1 (the red circle and the green triangle curves), indicating the synchronized state within one community. However, the phases of oscillators in different communities are different, even for those belong to the same sort (the blue up triangle and light blue diamond lines), and the order parameter for the whole network is much less than 1 (the black square line). The order parameter for each community is near to 1, for community sorts A and B, the

order parameter are almost equal and much lower than those for individual community, and the order parameter of the whole network is the lowest one. In the case $C = 0.1$, the oscillators in the same community will also synchronize soon. With the further increasing of coupling strength, an interesting phenomenon appears: oscillators belong to the same sort are synchronized before the appearance of global collective consensus. Although the communities are equal in structure, the intrinsic frequencies divide the communities into two groups. We call this phenomenon *Community Grouping*. In the case $C = 0.3$, when the coupling strength is not too large, the *Community Grouping* can also be observed, while for sufficiently large σ , this phenomenon disappears and the global synchronization is achieved. In figure 2, the averaged sine of phases of oscillators are plotted with time at steady states for community networks with $C = 0.3$. When $\sigma = 0.5$, the coupling is too weak to form any synchronization. When $\sigma = 3.5$, *Community Grouping* phenomenon appears. When the coupling strength is strong enough ($\sigma = 9.0$), all the communities rotate coherently. From figures 1 and 2 we know that only in some proper regions of (C, σ) , the *Community Grouping* phenomenon could appear. We also investigate the network with more than two community sorts, and find that if the numbers of communities contained by one sort are equal, the network exhibits similar synchronization properties to the case of two sorts.

As to networks with two sorts containing different numbers of communities, some interesting phenomena appears. Figure 3 shows the changing of the order parameters with the coupling strength, where sort A and sort B contain 3 and 7 communities, respectively. For different community strengths $C = 0.01, 0.1$ and 0.3 , order parameters for individual community all reach 1 soon, showing that oscillators within one community synchronize quickly. For the networks with very strong community structure ($C = 0.01$), even the communities belong to the same sort could not get synchronized each other. Meanwhile, for larger C , the order parameters for sort A and B are not equal: with the coupling strength increasing from 0, both M_A and M_B rise soon, but the former is faster. While, with the further increasing of the coupling strength, the rising of the two order parameter slow down, and the rising of M_B becomes a little faster, sooner or later M_B will surpass M_A . This crossing behavior of M_A and M_B can be observed in the panel with $C = 0.1$, and larger C will make the crossing point smaller. After the crossing point, M_A will increase, decrease and increase again, and finally reaches 1 (see the inset of Figure 3). The abnormal change of M_A can be simply explained as follows. When σ is small, with fewer oscillators, A will show better coherent. For larger σ , interactions between oscillators are strong, because of containing more community, B has stronger power of influence, thus oscillators in A are disturbed by B. This effect could enhance the global synchronization, but simultaneously reduce M_A .

When there are couplings between oscillators, the os-

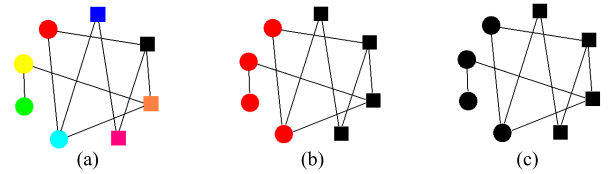


FIG. 6: (Color online) The states of the communities. The circle and the square present two sort of community and the communities with the same color indicate they are synchronized. (a), (b) and (c) represent the cases of $C = 0.01, 0.1$ and 0.3 , as shown in the above numerical simulations.

cillators will not rotate at their own frequencies ω but at actual frequencies Ω , and the distribution of Ω depends on the coupling strength and the structure of the networks. Figure 4 and 5 shows the relationships between actual frequencies and intrinsic frequencies of each oscillator at different coupling strength given $C = 0.3$. In figure 4, there are two sorts with equal number of communities. It can be seen that for a weaker coupling strength, the actual frequencies Ω_i are almost equal to the intrinsic frequencies ω_i , increasing the coupling strength σ will make the actual frequencies of oscillators on the same sort more consensus, till the oscillators rotate at a common frequency near their own average frequencies, indicated by the two parallel lines in the third panel. Further increasing the coupling strength will make the two parallel lines nearer and nearer, and finally, all the oscillators rotate at about the average frequency of the whole system, indicating the achievement of the global synchronization. Figure 5 reports the results of network consisted of two sorts with different numbers of communities. The actual frequencies will change alike, and the final synchronized frequency is about the average frequency 1.2.

IV. CONCLUSION AND DISCUSSION

In conclusion, we investigated the synchronization properties in complex networks with different sorts of communities and found that when the community structure is strong, only the oscillators in the same community synchronize, with the weakening of the community strength, oscillators in the same sort of communities will synchronize independently, only when the community structure is not evident that all the oscillators in the network can synchronize, this is clearly shown in figure 6. When the numbers of communities in different sorts are not equal, the sort with fewer communities shows better consensus at weaker coupling and when the couplings between oscillators are strong, the sort with more communities have larger order parameter. When all the oscillators are synchronized, they will rotate at the average frequency. The huge-size real-life networks are generally consisted of many communities, and those communities may be functional different. In this sense, to study the

detailed dynamical properties of network with functionally different communities is of interesting. The current work provides a start point on this issue, which is helpful for the in-depth understanding of synchronization process on networks.

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- [1] R. Albert and A. L. Barabási, *Rev. Mod. Phys.* **74**, 47(2002).
 - [2] M. Girvan, and M. E. J. Newnam, *Proc. Natl. Acad. Sci. U.S.A.* **99**, 7821(2002).
 - [3] G. Palla, I. Derényi, I. Farkas, and T. Vicsek, *Nature* **435**, 814(2005).
 - [4] S. H. Strogatz, *SYNC-How the emerges from chaos in the universe, nature, and daily life* (Hyperion, New York, 2003).
 - [5] D. J. Watts and S. H. Strogatz, *Nature* **393**, 440 (1998).
 - [6] A.-L. Barabási and R. Albert, *Science* **286**, 509 (1999).
 - [7] P. M. Gade and C.-K. Hu, *Phys. Rev. E* **62**, 6409 (2000).
 - [8] X. F. Wang and G. Chen, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **12**, 187 (2002).
 - [9] X. F. Wang, and G. Chen, *IEEE Trans. Circuits and Systems I* **49**, 54 (2002).
 - [10] M. Barahona and L. M. Pecora, *Phys. Rev. Lett.* **89**, 054101 (2002).
 - [11] T. Nishikawa, A. E. Motter, Y.-C. Lai, and F. C. Hoppensteadt, *Phys. Rev. Lett.* **91**, 014101 (2003).
 - [12] H. Hong, B. J. Kim, M. Y. Choi, and H. Park, *Phys. Rev. E* **69**, 067105 (2004).
 - [13] P. N. McGraw, M. Menzinger, *Phys. Rev. E* **72**, 015101 (2005).
 - [14] M. Zhao, T. Zhou, B.-H. Wang, G. Yan, H.-J. Yang, and W.-J. Bai, *Physica A*, **371**, 773 (2006).
 - [15] X. Wu, B.-H. Wang, T. Zhou, W.-X. Wang, M. Zhao, and H.-J. Yang, *Chin. Phys. Lett.* **23**, 1046 (2006).
 - [16] F. Sorrentino, M. di Bernardo, G. H. Cuéllar, and S. Boccaletti, *Physica D* **224**, 123 (2006).
 - [17] M. Chavez, D.-U. Hwang, J. Martinerie, and S. Boccaletti, *Phys. Rev. E* **74**, 066107 (2006).
 - [18] A. E. Motter, C. Zhou, and J. Kurths, *Phys. Rev. E* **71**, 016116 (2005).
 - [19] A. E. Motter, C. Zhou, and J. Kurths, *Europhys. Lett.* **69**, 334 (2005).
 - [20] A. E. Motter, C. Zhou, and J. Kurths, *AIP Conf. Proc.* **776**, 201 (2005).
 - [21] D.-U. Hwang, M. Chavez, A. Amann, and S. Boccaletti, *Phys. Rev. Lett.* **94**, 138701 (2005).
 - [22] M. Chavez, D. -U. Hwang, A. Amann, H. G. E. Hentschel, and S. Boccaletti, *Phys. Rev. Lett.* **94**, 218701 (2005).
 - [23] M. Zhao, T. Zhou, B.-H. Wang, and W.-X. Wang, *Phys. Rev. E* **72**, 057102 (2005).
 - [24] C. Zhou and J. Kurths, *Phys. Rev. Lett.* **96**, 164102 (2006).
 - [25] T. Zhou, M. Zhao, and B.-H. Wang, *Phys. Rev. E* **73**, 037101 (2006).
 - [26] C.-Y. Yin, W.-X. Wang, G. Chen, and B.-H. Wang, *Phys. Rev. E* **74**, 047102 (2006).
 - [27] M. Zhao, T. Zhou, B.-H. Wang, Q. Ou, and J. Ren, *Eru. Phys. J. B* **53**, 375 (2006).
 - [28] Q. Guo, J.-G. Liu, R.-L. Wang, X.-W. Chen, and Y.-H. Yao, *Chin. Phys. Lett.* **24**, 2437 (2007).
 - [29] X. Wang, Y.-C. Lai, and C. H. Lai, *Phys. Rev. E* **75**, 056205 (2007).
 - [30] Y.-F. Lu, M. Zhao, T. Zhou, and B.-H. Wang, *arXiv*: 0708.0863.
 - [31] M. Zhao, T. Zhou, G. Chen, and B.-H. Wang, *Front. Phys. China* **2**, 460 (2007).
 - [32] E. Oh, K. Rho, H. Hong, B. Kahng, *Phys. Rev. E* **72**, 047101 (2005).
 - [33] K. Park, Y.-C. Lai, S. Gupte, J.-W. Kim, *Chaos* **16**, 015105 (2006).
 - [34] L. Huang, K. Park, Y.-C. Lai, L. Yang, and K. Yang, *Phys. Rev. Lett.* **97**, 164101 (2006).
 - [35] T. Zhou, M. Zhao, G. Chen, G. Yan, and B. -H. Wang, *Phys. Lett. A* **368**, 431 (2007).
 - [36] M. E. J. Newman, *SIAM Review*, **45**, 167 (2003).
 - [37] Y. Kuramoto, in *Internaltional Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture notes in Physics No. 30 (Springer, New York, 1975).
 - [38] Y. Kuramoto, *Chemical Oscillations, Wave and Turbulence* (Springer-Verlag, Berlin, 1984).
 - [39] Y. Kuramoto, and I. Nishikawa, *J. Stat. Phys.* **49**, 569 (1987).
 - [40] A. Pikovsky, *Synchronization* (Cambridge University Press, Cambridge, 2001).
 - [41] J. A. Acebrón, L. L. Bonilla, C. J. P. Vicente, F. Ritort, and R. Spigler, *Rev. Mod. Phys.* **77**, 137 (2005).
 - [42] G. Yan, Z. -Q. Fu, J. Ren, and W. -X. Wang, *Phys Rev E* **75**, 016108 (2007).
 - [43] P. L. Krapivsky, S. Render, and F. Leyvraz, *Phys. Rev. Lett.* **85**, 4629(2000).
 - [44] T. Zhou, G. Yan, and B. -H. Wang, *Phys. Rev. E* **71**, 046141 (2005).